Table 1 K values for propellant evaluation

First stage	Second stage	Third stage	∞ stage
0.70	0.36	0.25	0.15

Table 2 I_s and ρ for three hypothetical propellants

Propellant	I_s, sec	ρ, g/cm³
A) low I_s, high ρ	268	1.85
B) high I_s, low ρ	280	1.60
C) references	275	1.70

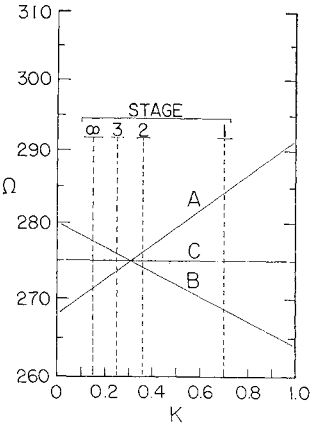

Fig. 1 Graphical propellant selection

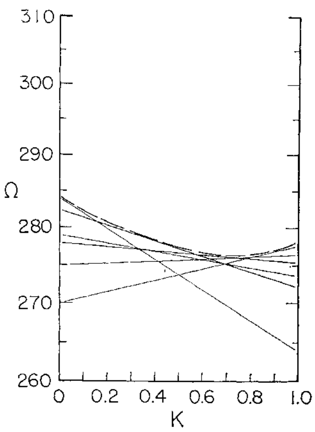

Fig. 2 Upper performance limits for a given type of propellant

line, a propellant with $\rho > \rho_0$ gives a line with a positive slope, and a propellant with $\rho < \rho_0$ gives a line with a negative slope.

Figure 1 shows the relative merits for each of the stages for three hypothetical propellants. Their specific impulses and densities are given in Table 2.

It is readily apparent that propellant A is the best of the three for the first and second stages, despite the fact that it has the lowest I_s. Propellant B is best only for the third stage, despite its high I_s. All three propellants would serve equally well for a stage with $K = 0.31$. Figure 1 also shows that, although the effect of density is smaller the higher the stage, density must be considered even for an ∞ stage.

These charts also can be used to select the optimum propellant composition or mixture ratio for each stage. The effective specific impulse Ω may vary with composition for a given type of solid propellant as shown in Fig. 2. The straight lines define an envelope showing the performance limits for this type of propellant; the optimum composition for each stage can be approximated by interpolation.

Reference

[1] Gordon, L. J., "Trade-off between propellant specific impulse and density," Aerospace Eng. **20**, 12–13, 27 (November 1961).

Direct Nonlinear Stability Analysis of Keplerian Orbital Motion

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THE practical interest in and usefulness of Keplerian orbits is bound up inextricably with their positive stability characteristics: ubiquitous small forces and deviations that never can be accounted fully do not alter totally the character of the idealized motion. Viewing these disturbance motions as oscillatory perturbations superimposed on a basic orbit is frequently justified by the convenience, although it is strictly more accurate to consider that each disturbance gives rise to an altogether new orbit. The customary "first-approximation stability" analysis performed in this manner not only is tedious in cases of practical interest but also suffers from the more serious objection of being sometimes totally misleading.[1] The modern form of ideas first formulated by H. Poincaré permits the investigation of the stability of orbital motion in a rigorous and yet eminently practical fashion.

It is a familiar elementary exercise to be found in textbooks on dynamics to show that circular orbits possess first-approximation stability against small disturbances when an inverse-square force field is assumed. The conclusion is based on the known character of solutions of a familiar differential equation that only approximately describes the disturbance motion. For actual elliptic orbits of nonzero eccentricity, on the other hand, the same techniques lead to Mathieu equations and a problem of so much greater difficulty that none of the standard treatises contain its discussion. On purely physical qualitative grounds the omission can be excused, since the property established for a circular orbit seems plausible also for slightly eccentric ellipses, at least. For problems of satellite attitude control, however, in which librations and orbital oscillations occur and interact in near-resonance with each other, more accurate results are needed.[2] It is shown how a Liapunov function can be constructed from the energy integral of orbital motion, and hence stability is proved. An essential feature is that the conclusion does not depend on the nature of the solutions of the relevant differential equations (or even some approximation to these); the rigorous nonlinear equations themselves furnish the required results directly.

Orbital motion is specified by a pair of second-order ordinary differential equations, one of which is immediately integrable and introduces the area parameter h:

$$\ddot{r} - r\dot{\theta}^2 = -G/r^2$$

$$r^2\dot{\theta} = h$$

(A common notation is employed, r,θ denoting plane polar position coordinates of the satellite centroid, dots denoting time derivatives, and G denoting the constant of earth-gravitation). The corresponding energy integral per unit mass is

$$E = [(\dot{r}^2 + r^2\dot{\theta}^2)/2] - (G/r)$$

Consider a "basic" orbit $R(t)$, $\vartheta(t)$, and superposed on it a disturbed motion, the whole being given by

$$r = R + \rho$$

$$\theta = \vartheta + \varphi$$

where ρ, $\dot{\rho}$, and $\dot{\varphi}$ are small perturbations, and simplify the

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equations that follow by assuming a circular orbit (so that $\dot{R} = \ddot{R} = 0$). Then by adding to the constant energy E a quantity corresponding to its value in the basic orbit, $-G/2R$, a function $V(\rho,\dot{\rho})$ is obtained in the form

$$V(\rho,\dot{\rho}) = E + \frac{G}{2R} = \frac{\dot{\rho}^2}{2} + \frac{G}{R^3}\frac{\rho^2}{2}$$

when all third and higher powers of ρ and $\dot{\rho}$ are neglected.

The function V so defined has the following properties: 1) $V(\rho,\dot{\rho})$ and its first partial derivatives are continuous in a region that includes the point $\rho = \dot{\rho} = 0$; 2) $V(0, 0) = 0$, an isolated minimum; and 3) $V > 0$ in a region that excludes only the point (0, 0), and the function is positive definite. Any function that satisfies these conditions and, in addition, is such that $\dot{V} \leqq 0$ is called a Liapunov function. It is verified easily that in this case $\dot{V} = 0$, identically, in fact, by noticing that

$$\dot{V} = \dot{\rho}\cdot[\ddot{\rho} + (G/R^3)\rho] = 0$$

the term in brackets being exactly the derivative of the (constant) energy of the motion. Under these circumstances, Chetayev's proofs are applicable and establish that the motion is stable.[1] It is noted further that, because $-\dot{V}$ is *not* positive definite, the motion also is not asymptotically stable (that is, the disturbance does not ultimately decay and restore the "basic" orbit).

Analogous results completely are obtained also for orbits of nonzero eccentricity with these relatively minor differences of detail: that the Liapunov function contains a greater number of terms corresponding to the fact that the eccentricity is nonzero, and that the angular velocity no longer is suppressed so easily as in the present case by introduction of the constant h. Even when librational interactions with orbital disturbances raise the order of the differential equation system, the same procedure exactly avails with no essential complication or loss of rigor.

References

[1] Chetayev, N. G., *The Stability of Motion* (Pergamon Press Inc., New York, 1961), Chap. 2.

[2] Michelson, I., "Orbit-resonance of gravity-gradient satellites," AIAA J. **1**, 489–490 (1963).

Approximate Longitudinal Dynamics of a Lifting Orbital Vehicle

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IN Ref. 1 an analysis is made of the dynamic longitudinal stability of a vehicle on an orbital path. Analysis of the mode shapes showed that, for two of the modes, the phugoid and the arrow, the angle of attack varied only slightly. It is of interest to compare the numerical results of Ref. 1 with those obtained from an analysis in which the a priori assumption of constant angle of attack is made.

The derivation of the equations for constant angle of attack is straightforward, with those given in Ref. 2 being applicable to this case also. Comparisons between the results from the two methods are presented in Figs. 1–3. The approximate results are within about 30% of the exact results for the phugoid period, with somewhat more deviation obtained for the phugoid damping. These results are about what should be expected, based upon the known accuracy of the approximation for the classical phugoid. The characteristics of the arrow mode are predicted within a factor of about 2. Although the numerical accuracy is not particularly high, the results do give fairly good comparison over a very wide range of conditions. The results, therefore, should be adequate for preliminary estimates.

The details are given in Ref. 3, along with generalized charts based upon the same approximation.

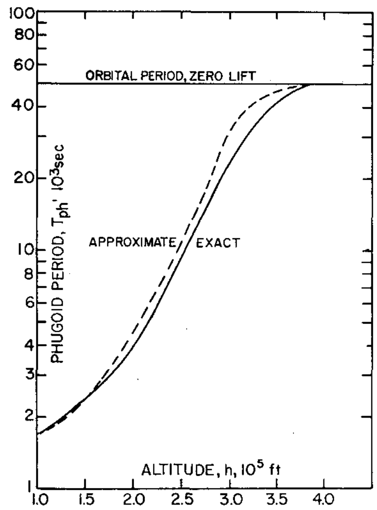

Fig. 1 Comparison of results, phugoid period

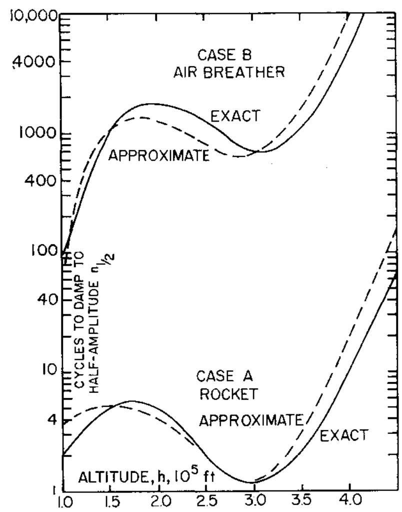

Fig. 2 Comparison of results, phugoid damping

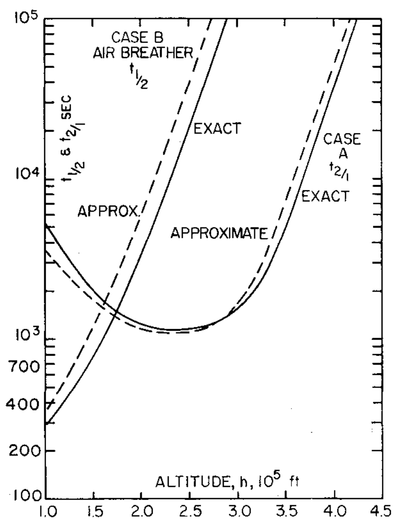

Fig. 3 Comparison of results, "arrow" mode

References

[1] Etkin, B., "Longitudinal dynamics of a lifting vehicle in orbital flight," J. Aerospace Sci. **28**, 779–788, 832 (1961).

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